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1998 J. Phys. A: Math. Gen. 31 1253

(http://iopscience.iop.org/0305-4470/31/4/014)

J. Phys. A: Math. Gen. **31** (1998) 1253–1266. Printed in the UK PII: S0305-4470(98)85415-2

Homology and cohomology on generalized Poisson manifolds

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Received 27 June 1997

Abstract. The canonical homology of a generalized Poisson manifold is introduced, and the two spectral sequences associated with the periodic double complex are studied. The generalized Poisson cohomology is also discussed, and a natural parity with the canonical homology is established. The results hold for Nambu–Poisson manifolds.

1. Introduction

The study of multibrackets on manifolds has recently gained an increasing interest due to the geometric formulation of Takhtajan [23] of Nambu–Poisson brackets. A Nambu–Poisson bracket is a multibracket enjoying an integrability property which is a natural generalization of Jacobi's identity for Poisson brackets. Another different kind of multibrackets (called generalized Poisson) were recently introduced by Azcárraga *et al* [1, 2] (see also [3]). They are brackets of even order, and enjoy a very different (in principle) integrability condition which is the natural extension of the vanishing of the Schouten–Nijenhuis bracket of the multivector Λ defining the multibracket. This notion is meaningful provided Λ is of an even order, otherwise $[\Lambda, \Lambda]$ trivially vanishes. Recently [15], we introduced a more general kind of geometrical structure which permits us to simultaneously discuss both types of brackets. The point is to forget for a while the integrability condition. So, we are in the presence of a multivector Λ of order, say n, on a manifold M. Λ will be called a generalized almost Poisson structure and M a generalized almost Poisson manifold. The relationship between the algebra level (the multibracket) and the manifold level (the multivector) is given by the formula

$$\{f_1, \ldots, f_n\} = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_n).$$

The purpose of this paper is to introduce and study the canonical homology and relate it to the generalized Poisson cohomology of a generalized Poisson structure. For a generalized Poisson manifold (M, Λ) the canonical homology is obtained by defining a

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0305-4470/98/041253+14$19.50

Koszul differential $\delta = [i(\Lambda), \mathrm{d}]$ which extends the well known operator defined by Koszul for Poisson manifolds. If Λ is of order $2n$ we get $2n-1$ canonical complexes which define the so-called canonical homology of M. Since $\mathrm{d}\delta + \delta\mathrm{d} = 0$, we can also consider the periodic double complex $C^{\mathrm{per}}_{p,q}(M) = \Omega^{(2n-1)q-p}(M)$, $p, q \in \mathbb{Z}$, where d is the horizontal differential and δ is the vertical differential. This double complex is the natural extension of that defined by Brylinski [6]. This fact leads us to discuss the following two problems.

Problems.

(i) Give conditions on a compact generalized Poisson manifold M which ensure that any de Rham cohomology class has a harmonic (with respect to the generalized Poisson structure) representative α, that is, $\mathrm{d}\alpha = \delta\alpha = 0$.

(ii) Study the degeneration of the two spectral sequences associated to the generalized Poisson structure.

In view of the first problem, we know that it holds for compact symplectic manifolds satisfying the hard Lefschetz theorem [21], but it is not true for arbitrary compact symplectic manifolds [9]. In view of the second problem, in section 3 we prove that the second spectral sequence always degenerates at the first term, but it is not the case for the first spectral sequence as we have seen for Poisson manifolds [10]. In section 3, we also discuss several interesting examples of generalized Poisson structures: volume forms, hyper-Kähler and bi-Hamiltonian manifolds.

In section 4 we define a contravariant differentiation on the algebra of multivectors by $\partial(P) = -[P, \Lambda]$ which yields a family of $2n-1$ complexes. The corresponding cohomology is called the generalized Poisson cohomology, and extends the one defined for Poisson manifolds [24]. If G is a Lie group endowed with a left invariant generalized Poisson structure, then it is an obvious relation between the generalized Poisson cohomology restricted to the left invariant multivectors on G and the generalized Poisson cohomology discussed in [3] (see also [22]). We also obtain a natural parity between the canonical homology and the generalized Poisson cohomology of a generalized Poisson manifold.

We finally remark that a Nambu–Poisson manifold of even order is also generalized Poisson, so the above results hold for Nambu–Poisson manifolds of even order. More than this, all the constructions and results remain valid when one considers almost generalized Poisson structures of odd order, since in this case $[\Lambda, \Lambda]$ always vanishes. This permits us to apply our results to arbitrary Nambu–Poisson manifolds (of odd or even order, indistinctly) (see section 5).

2. Generalized Poisson manifolds

Let M be a differentiable manifold of dimension m. We denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M and by $C^\infty(M, \mathbb{R})$ the algebra of C^∞ real-valued functions on M.

An *almost Poisson n-tensor* on M is a skew-symmetric tensor Λ of type $(n, 0)$ (see [15]). If Λ is an almost Poisson n-tensor on M then we can define a n-linear mapping on $C^\infty(M, \mathbb{R})$ as follows

$$\{f_1, \ldots, f_n\} = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_n) \qquad \text{for } f_1, \ldots, f_n \in C^\infty(M, \mathbb{R}). \tag{1}$$

The bracket $\{, \ldots, \}$ is skew-symmetric and satisfies the Leibniz rule. Thus, it is an *almost Poisson bracket of order n*. Conversely, if $\{, \ldots, \}$ is an almost Poisson bracket of order n then an almost Poisson tensor Λ can be defined by (1) (see [15]). In such a case, (M, Λ) is called a *generalized almost Poisson manifold*. If $\Omega^r(M)$ is the space of r-forms on M then the n-vector Λ induces a $C^\infty(M, \mathbb{R})$-linear mapping $\# : \Omega^{n-1}(M) \longrightarrow \mathfrak{X}(M)$ given by

$$\langle \#(\alpha_1 \wedge \ldots \wedge \alpha_{n-1}), \beta \rangle = \Lambda(\alpha_1, \ldots, \alpha_{n-1}, \beta) \qquad \text{for } \alpha_1, \ldots, \alpha_{n-1}, \beta \in \Omega^1(M). \tag{2}$$

Therefore, if $f_1, \ldots, f_{n-1}$ are $n-1$ functions on M, we define a vector field $X_{f_1 \ldots f_{n-1}} = \#(\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_{n-1})$ which is called the *Hamiltonian vector field* associated with the Hamiltonian functions $f_1, \ldots, f_{n-1}$.

A richer structure, related to interesting dynamical problems, can be considered by adding integrability conditions to the almost Poisson bracket. Two, in principle, unrelated integrability conditions may be assumed.

(3a) (*Generalized Jacobi identity*); for n even,

$$\mathrm{Alt}\,(\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\}) = 0$$

for all functions $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n$ on M. This is equivalent to $[\Lambda, \Lambda] = 0$. In this case, $\{, \ldots, \}$ (resp. Λ) is called a *generalized Poisson bracket* (resp. *tensor*) and (M, Λ) is a *generalized Poisson manifold* [1, 2, 15].

(3b) (*Fundamental identity*)

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_i\} \ldots, g_n\}$$

for all functions $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n$ on M. In this case, $\{, \ldots, \}$ (resp. Λ) is called a *Nambu–Poisson bracket* (resp. *tensor*) and M is a *Nambu–Poisson manifold* [15, 23].

Notice that if $n = 2$ then (3a) and (3b) are equivalent and M is a Poisson manifold. On the other hand, an even-order Nambu–Poisson structure is generalized Poisson [15].

Now, let M be an oriented n-dimensional manifold and v_M a volume form. Given n functions $f_1, \ldots, f_n$ on M, we define its bracket by the formula

$$\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n = \{f_1, \ldots, f_n\} v_M.$$

It is not hard to prove that it is a Nambu–Poisson bracket [13]. Moreover, a Nambu–Poisson tensor $\Lambda \neq 0$ of order n comes from a volume form. More generally, the Hamiltonian vector fields on a Nambu–Poisson manifold of order $n \geqslant 3$ generate a generalized foliation $\mathcal{D}$ whose leaves are either points or n-dimensional Nambu–Poisson manifolds with Nambu–Poisson structure coming from a volume form [15] (see also [13, 20, 25]).

3. The canonical complex

This section is devoted to the generalization of the canonical homology and the related spectral sequences studied in [6, 9, 10–12, 14] for Poisson manifolds.

Definition 3.1. Let (M, Λ) be a generalized Poisson manifold of order $2n$. Then we define the *generalized Koszul (or canonical) differential* δ as the commutator of $i(\Lambda)$ and the exterior differential d, that is,

$$\delta = [i(\Lambda), \mathrm{d}] = i(\Lambda) \circ \mathrm{d} - \mathrm{d} \circ i(\Lambda)$$

where $i(\Lambda)$ denotes the inner product by Λ.

So, we have

$$\delta : \Omega^k(M) \longrightarrow \Omega^{k-2n+1}(M)$$

for $k \geqslant 2n - 1$; and $\delta\alpha = 0$, for $\alpha \in \Omega^k(M)$, with $k \leqslant 2n - 2$.

Proposition 3.2. Let (M, Λ) be a generalized Poisson manifold of order $2n$. Then

$$\begin{aligned}\delta(f_0 \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k) = &\sum_{1\leqslant i_1<\ldots<i_{2n-1}\leqslant k} (-1)^{i_1+\ldots+i_{2n-1}+n} \\ &\times\{f_0, f_{i_1}, \ldots, f_{i_{2n-1}}\} \mathrm{d}f_1 \wedge \ldots \wedge \widehat{\mathrm{d}f_{i_1}} \wedge \ldots \wedge \widehat{\mathrm{d}f_{i_{2n-1}}} \wedge \ldots \wedge \mathrm{d}f_k \\ &+ \sum_{1\leqslant i_1<\ldots<i_{2n}\leqslant k} (-1)^{i_1+\ldots+i_{2n}+n+1} f_0 \\ &\times \mathrm{d}\{f_{i_1}, \ldots, f_{i_{2n}}\} \wedge \mathrm{d}f_1 \wedge \ldots \wedge \widehat{\mathrm{d}f_{i_1}} \wedge \ldots \wedge \widehat{\mathrm{d}f_{i_{2n}}} \wedge \ldots \wedge \mathrm{d}f_k.\end{aligned}$$

Proof. The results follows from a direct calculation using the definition of the generalized Koszul differential. □

Lemma 3.3. We have

(i) $2i(\Lambda)di(\Lambda) = \mathrm{d}i(\Lambda)^2 + i(\Lambda)^2\mathrm{d}$;

(ii) $ki(\Lambda)\mathrm{d}i(\Lambda)^{k-1} = (k-1)\mathrm{d}i(\Lambda)^k + i(\Lambda)^k\mathrm{d}, \forall k \in \mathbb{N}$.

Proof. A general property of the Schouten–Nijenhuis bracket is that $[[i(P), \mathrm{d}], i(Q)] = i([P, Q])$. Therefore, taking $P = Q = \Lambda$, we have $[[i(\Lambda), \mathrm{d}], i(\Lambda)] = i([\Lambda, \Lambda]) = 0$, which implies $2i(\Lambda)\mathrm{d}i(\Lambda) = \mathrm{d}i(\Lambda)^2 + i(\Lambda)^2\mathrm{d}$. This proves (i). Part (ii) follows by induction. □

Now, from lemma 3.3 (part (i)), we deduce that $\delta^2 = 0$. So, we have the family of $2n-1$ canonical complexes

$$\ldots \longrightarrow \Omega^{2(2n-1)+j}(M) \stackrel{\delta}{\longrightarrow} \Omega^{(2n-1)+j}(M) \stackrel{\delta}{\longrightarrow} \Omega^j(M) \longrightarrow 0$$

for $j = 0, \ldots, 2n-2$. The jth complex defines a homology which is called the *j-canonical homology* of M. The homology group of degree $k = i(2n-1) + j$ is denoted by $H_k^{j\,\mathrm{can}}(M)$. It is clear that if M is a Poisson manifold then $H_*^{0\,\mathrm{can}}(M)$ is just the canonical homology $H_*^{\mathrm{can}}(M)$ of M studied by Brylinski [6].

Example 3.4 (volume forms). Let υ_M be a volume form in a compact (connected) manifold M of dimension $2n \geqslant 4$. We obtain the following two non-trivial complexes

$$\Omega^{2n-1}(M) \stackrel{\delta}{\longrightarrow} \Omega^0(M) \longrightarrow 0$$
$$\Omega^{2n}(M) \stackrel{\delta}{\longrightarrow} \Omega^1(M) \longrightarrow 0.$$

The other complexes are trivial. Therefore, the canonical homology groups are:

$$\begin{aligned}&H_0^{0\,\mathrm{can}}(M) = \Omega^0(M)/\delta(\Omega^{2n-1}(M)) \cong \mathbb{R}\\ &H_{2n-1}^{0\,\mathrm{can}}(M) = \{\alpha \in \Omega^{2n-1}(M) | \delta\alpha = 0\} = \{\alpha \in \Omega^{2n-1}(M) | \mathrm{d}\alpha = 0\}\\ &H_1^{1\,\mathrm{can}}(M) = \Omega^1(M)/\delta(\Omega^{2n}(M)) = \Omega^1(M)/\{\mathrm{d}f | f \in C^\infty(M, \mathbb{R})\}\\ &H_{2n}^{1\,\mathrm{can}}(M) = \{\alpha \in \Omega^{2n}(M) | \delta\alpha = 0\} \cong \mathbb{R}\\ &H_k^{j\,\mathrm{can}}(M) = \Omega^k(M) \qquad \text{for } k = 2, \ldots, 2n-2,\ j \neq 0, 1.\end{aligned}$$

Remark 3.5. Example 3.4 shows that, in general, the canonical homology groups are not finite dimensional.

A straightforward computation shows the following.

Proposition 3.6. We have

$$d\delta + \delta d = 0.$$

From proposition 3.6 we may define the canonical double complex given by $C_{p,q}(M) = \Omega^{(2n-1)q-p}(M)$, for $p, q \geqslant 0$. We also can define the periodic double complex $C^{\mathrm{per}}_{p,q}(M) = \Omega^{q(2n-1)-p}(M)$, for $p, q \in \mathbb{Z}$, which has d for horizontal differential and δ for vertical differential, both of degree -1. More precisely, the vertical arrows are given by the $2n-1$ complexes, and they are horizontally connected by the exterior differential.

Associated with the periodic double complex there are two spectral sequences $\{E^r(M)\}$ and $\{'E^r(M)\}$ that converge to the total homology $H^D_*(M)$ of the total complex, that is,

$$(C^{\mathrm{per}}_k(M) = \oplus_{p+q=k} C^{\mathrm{per}}_{p,q}(M), D = d + \delta).$$

Notice that the first term of the first spectral sequence is the canonical homology, that is, $E^1_{p,q}(M)$ is the canonical homology group of degree $q(2n-1) - p$. The first differential is $\delta_1 = d$. Moreover, the first term of the second spectral sequence is the de Rham cohomology: $'E^1_{p,q}(M) = H^{q(2n-1)-p}_{dR}(M)$, and the first differential is $'\delta_1 = \delta$.

Next, we will discuss the following two problems, which are a natural extension of the corresponding ones posed by Brylinski [6] in the context of Poisson manifolds.

Problems.

(i) Give conditions on a compact generalized Poisson manifold M which ensure that any de Rham cohomology class has a harmonic (with respect to the generalized Poisson structure) representative α, that is, $d\alpha = \delta\alpha = 0$.

(ii) Study the degeneration of the two spectral sequences associated to the generalized Poisson structure.

Theorem 3.7. For any generalized Poisson manifold (M, Λ), the second spectral sequence of the double complex $C^{\mathrm{per}}_{p,q}(M)$ degenerates at the first term $'E^1(M)$, that is, $'E^1(M) \cong {}'E^\infty(M)$.

Proof. This follows from lemma 3.3 and the definition of the second spectral sequence, in a similar way that for the Poisson case [10]. □

Remark 3.8. The above two problems have already been studied for compact Poisson manifolds. For compact symplectic manifolds it was proved by Mathieu [21] that any de Rham cohomology class has a symplectically harmonic representative if and only if it satisfies the hard Lefschetz theorem. So, in particular the result is true for compact Kähler manifolds [6]. However, the result does not hold for arbitrary symplectic manifolds [9]. For compact cosymplectic manifolds the result holds, but not for compact almost cosymplectic manifolds [14].

In view of the second problem, we have that the first spectral sequence degenerates at the first term for compact symplectic manifolds [6], but not for compact almost cosymplectic manifolds [10, 12], and then, not for compact Poisson manifolds. The second spectral sequence always degenerates at the first term [10, 12].

We also notice that the canonical double complex was recently extended for Jacobi manifolds by Chinea *et al* [7, 8].

Example 3.9 (volume forms). Using the results in example 3.4, we deduce that for a generalized Poisson structure coming from a volume form on a compact (connected) manifold M any de Rham cohomology class has a harmonic (with respect to the generalized Poisson structure) representative. Moreover, the first spectral sequence degenerates at the second term, that is, $E^2(M) \cong E^\infty(M)$.

Example 3.10 (hyper-Kähler manifolds). Some interesting examples of generalized Poisson manifolds are the hyper-Kähler manifolds [5, 16], that is, differentiable manifolds of dimension $4n$ with a Riemannian metric g and three complex structures J_1, J_2, J_3 compatible with g and such that:

(i) the complex structures satisfy the quaternionic relations, i.e. $J_3 = J_1 J_2 = -J_2 J_1$;

(ii) the Kähler forms ω_i defined by $\omega_i(X, Y) = g(X, J_i Y)$, for $X, Y \in \mathfrak{X}(M)$, are closed.

If we take the Poisson structures Λ_i associated to the Kähler forms ω_i, it can be proved that they are compatible (i.e. $[\Lambda_i, \Lambda_j] = 0, \forall i, j$). Therefore, the fundamental 4-vector

$$\Lambda = \Lambda_1 \wedge \Lambda_1 + \Lambda_2 \wedge \Lambda_2 + \Lambda_3 \wedge \Lambda_3 \tag{3}$$

satisfies $[\Lambda, \Lambda] = 0$, so it is a generalized Poisson structure of order 4 on M.

Let δ be the generalized Koszul operator on M. Using (3) and lemma 3.3, we deduce that

$$\delta = 2 \sum_{j=1}^{3} (i(\Lambda_j)\delta_j) \tag{4}$$

where δ_j is the Koszul operator associated to the Poisson 2-vector Λ_j.

On the other hand, from the results of Brylinski [6] (see corollary 2.4.2 in [6]), we obtain that if (N, J, g) is a Kähler manifold and α is a harmonic form on N with respect to the Riemannian metric g then α is also harmonic with respect to the associated Poisson structure on N. Using this fact and (4), we conclude that problem (i) has an affirmative answer for the generalized Poisson structure of a compact hyper-Kähler manifold, that is, if M is a compact hyper-Kähler manifold then any de Rham cohomology class has a harmonic representative with respect to the generalized Poisson structure.

Example 3.11 (bi-Hamiltonian manifolds). Let Λ_1 and Λ_2 be two compatible Poisson structures on a m-dimensional manifold M. Then $\Lambda = \Lambda_1 \wedge \Lambda_2$ is a generalized Poisson manifold of order 4 (see [16]). A simple computation shows that

$$\delta = i(\Lambda_1)\delta_2 + \delta_1 i(\Lambda_2)$$

where δ (respectively, δ_1 and δ_2) is the Koszul operator for Λ (respectively, Λ_1 and Λ_2).

Now, using the fact that $\delta_1 i(\Lambda_2) - i(\Lambda_2)\delta_1 = [[i(\Lambda_1), \mathrm{d}], i(\Lambda_2)] = i([\Lambda_1, \Lambda_2]) = 0$, it follows that

$$\delta = i(\Lambda_1)\delta_2 + i(\Lambda_2)\delta_1. \tag{5}$$

We consider the two following cases.

(i) Suppose that $\Lambda_1 = \Lambda_2 = \bar{\Lambda}$ and that $\bar{\Lambda}$ is the Poisson tensor of a symplectic form ω on M. Then, if M is compact and satisfies the hard Lefschetz theorem we deduce that problem (i) has an affirmative answer for Λ.

(ii) Suppose that M is compact, that g is a Riemannian metric on M and that (J_1, g_1) and (J_2, g_2) are two Kähler structures on M such that Λ_1 and Λ_2 are the Poisson tensors associated with the Kähler forms ω_1 and ω_2 of the structures (J_1, g) and (J_2, g) respectively. Then, using (5) and proceeding as in the case of a hyper-Kähler manifold, we conclude that problem (i) also has an affirmative answer for Λ.

4. Generalized Poisson cohomology

In this section we will extend the Chevalley–Eilenberg cohomology of the Lie algebra of the functions on Poisson manifolds to the framework of generalized Poisson manifolds. Then, the generalized Poisson cohomology can be seen as the cohomology of a family of subcomplexes of the generalized Chevalley–Eilenberg complexes.

Let (M, Λ) be a generalized Poisson manifold of order $2n$. We consider the vector space $C^k_{\mathrm{gCE}}(M)$ given by

$$C^k_{\mathrm{gCE}}(M) = \{\tilde{P} : C^\infty(M,\mathbb{R}) \times \overset{k}{\cdots} \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R}) | \tilde{P} \text{ is } k\text{-linear and skew-symmetric}\}.$$

Define the linear differential operator $\tilde{\partial} : C^k_{\mathrm{gCE}}(M) \longrightarrow C^{k+2n-1}_{\mathrm{gCE}}(M)$ by

$$\begin{aligned}\tilde{\partial}\tilde{P}(f_1,\ldots,f_{k+2n-1}) = &\sum_{1\leqslant i_1<\cdots<i_{2n}\leqslant k+2n-1} (-1)^{i_1+\cdots+i_{2n}+n+1} \\ &\times \tilde{P}(\{f_{i_1},\ldots,f_{i_{2n}}\}, f_1,\ldots,\widehat{f_{i_1}}\ldots,\widehat{f_{i_{2n}}},\ldots,f_{k+2n-1}) \\ &+ \sum_{1\leqslant i_1<\cdots<i_{2n-1}\leqslant k+2n-1} (-1)^{i_1+\cdots+i_{2n-1}+n+1} \\ &\times \{\tilde{P}(f_1,\ldots,\widehat{f_{i_1}},\ldots,\widehat{f_{i_{2n-1}}},\ldots,f_{k+2n-1}), f_{i_1},\ldots,f_{i_{2n-1}}\} \end{aligned} \tag{6}$$

for $\tilde{P} \in C^k_{\mathrm{gCE}}(M)$ and $f_1,\ldots,f_{k+2n-1} \in C^\infty(M,\mathbb{R})$.

Using the properties of the generalized Poisson bracket $\{,\ldots,\}$ we have that $\tilde{\partial}^2 = 0$. Hence, we may consider the family of $2n-1$ differential complexes

$$0 \longrightarrow C^j_{\mathrm{gCE}}(M) \xrightarrow{\tilde{\partial}} C^{(2n-1)+j}_{\mathrm{gCE}}(M) \xrightarrow{\tilde{\partial}} C^{2(2n-1)+j}_{\mathrm{gCE}}(M) \longrightarrow \ldots$$

for $j = 0,\ldots,2n-2$. The cohomology defined by these complexes will be called the *generalized Chevalley–Eilenberg cohomology* of M. The generalized Chevalley–Eilenberg cohomology group of degree $k = i(2n-1)+j$ of the complex j is denoted by $H^k_{j\,\mathrm{gCE}}(M)$. It is clear that if M is a Poisson manifold then $H^*_{0\,\mathrm{gCE}}(M)$ is just the Chevalley–Eilenberg cohomology of the Lie algebra of the functions on M (see [19]).

Now, let $\mathcal{V}^k(M)$ be the space of k-vectors on M and $\iota : \mathcal{V}^k(M) \longrightarrow C^k_{\mathrm{gCE}}(M)$ be the monomorphism of real vector spaces given by

$$\iota(P)(f_1,\ldots,f_k) = P(\mathrm{d}f_1,\ldots,\mathrm{d}f_k)$$

for $P \in \mathcal{V}^k(M)$ and $f_1,\ldots,f_k \in C^\infty(M,\mathbb{R})$. Notice that if $\tilde{P} \in C^k_{\mathrm{gCE}}(M)$ then $\tilde{P} \in \iota(\mathcal{V}^k(M))$ if and only if

$$\tilde{P}(f_1g_1, f_2,\ldots,f_k) = f_1\tilde{P}(g_1,f_2,\ldots,f_k) + g_1\tilde{P}(f_1,f_2,\ldots,f_k)$$

for $f_1, g_1, f_2,\ldots,f_k \in C^\infty(M,\mathbb{R})$.

Moreover, we have the following.

Proposition 4.1. If $P \in \mathcal{V}^k(M)$ then

$$\tilde{\partial}(\iota(P)) = \iota(\partial(P))$$

where $\partial : \mathcal{V}^k(M) \longrightarrow \mathcal{V}^{k+2n-1}(M)$ is the linear differential operator defined by

$$\partial(Q) = -[Q, \Lambda]$$

for $Q \in \mathcal{V}^k(M)$.

Proof. Using the characterization of the Schouten–Nijenhuis bracket given by Bhaskara and Viswanath [4] we have that

$$\begin{aligned}\partial P(\mathrm{d}f_1,\dots,\mathrm{d}f_{k+2n-1}) = &\sum_{1\leqslant i_1<\dots<i_{2n}\leqslant k+2n-1}(-1)^{i_1+\dots+i_{2n}+n+1}\\ &\times P(\mathrm{d}\{f_{i_1},\dots,f_{i_{2n}}\},\mathrm{d}f_1,\dots,\widehat{\mathrm{d}f_{i_1}},\dots,\widehat{\mathrm{d}f_{i_{2n}}},\dots,\mathrm{d}f_{k+2n-1})\\ &+\sum_{1\leqslant i_1<\dots<i_{2n-1}\leqslant k+2n-1}(-1)^{i_1+\dots+i_{2n-1}+n+1}\\ &\times\{P(\mathrm{d}f_1,\dots,\widehat{\mathrm{d}f_{i_1}},\dots,\widehat{df_{i_{2n-1}}},\dots,\mathrm{d}f_{k+2n-1}),f_{i_1},\dots,f_{i_{2n-1}}\}\end{aligned}$$

for $f_1,\dots,f_{k+2n-1}\in C^\infty(M,\mathbb{R})$.

Thus, from (6), we deduce the result. □

Definition 4.2. Let (M,Λ) be a generalized Poisson manifold of order $2n$. The linear differential operator $\partial:\mathcal{V}^k(M)\longrightarrow\mathcal{V}^{k+2n-1}(M)$ defined by

$$\partial P = -[P,\Lambda]$$

is called the *contravariant exterior differentiation*.

The contravariant exterior differentiation on a generalized Poisson manifold (M,Λ) was studied in [2] and the following result was proved.

Proposition 4.3 ([2]).

(i) $\partial^2=0$;

(ii) $\partial(P_1\wedge P_2)=(\partial P_1)\wedge P_2+(-1)^{\deg(P_1)}P_1\wedge(\partial P_2)$;

(iii) $\partial[P_1,P_2]=-[\partial P_1,P_2]-(-1)^{\deg(P_1)}[P_1,\partial P_2]$.

Notice that (i) follows directly from proposition 4.1.

Propositions 4.1 and 4.3 allow us to introduce the family of $2n-1$ differential subcomplexes of the generalized Chevalley–Eilenberg complexes of M:

$$0\longrightarrow\mathcal{V}^j(M)\xrightarrow{\partial}\mathcal{V}^{(2n-1)+j}(M)\xrightarrow{\partial}\mathcal{V}^{2(2n-1)+j}(M)\longrightarrow\dots$$

for $j=0,\dots,2n-2$. The cohomology defined by these complexes was called the *generalized Poisson* (gP) *cohomology* of M in [2]. The gP-cohomology group of degree $k=i(2n-1)+j$ is denoted by $H^k_{j\,\mathrm{gP}}(M)$. It is clear that if M is a Poisson manifold then the gP-cohomology of M is just the Lichnerowicz–Poisson (LP) cohomology defined by Lichnerowicz [19].

Remark 4.4.

(i) $H^0_{0\,\mathrm{gP}}(M)$ is the space of Casimir functions on M, that is, those functions c such that $\{f_1,\dots,f_{2n-1},c\}=0$ for all differentiable functions $f_1,\dots f_{2n-1}$.

(ii) If $n\geqslant 2$ then $H^1_{1\,\mathrm{gP}}(M)=\{X\in\mathfrak{X}(M)|L_X\Lambda=[X,\Lambda]=0\}$, that is, $H^1_{1\,\mathrm{gP}}(M)$ is the space of infinitesimal automorphisms of the generalized Poisson structure Λ.

(iii) $H^{2n}_{1\,\mathrm{gP}}(M)$ has a distinguished element $[\Lambda]$ defined by the $2n$-cocycle Λ. If $[\Lambda]=0$, M is called an *exact generalized Poisson manifold*.

Next, we will introduce a n-bracket of 1-forms and we will obtain the relation between this n-bracket and the contravariant exterior differentiation.

Let (M,Λ) be a generalized almost Poisson manifold of order n. We define an $\mathbb{R}$-multilinear, skew-symmetric operation

$$\{,\dots,\}:\Omega^1(M)\times\dots\times\Omega^1(M)\longrightarrow\Omega^1(M)$$

by

$$\{\alpha_1, \ldots, \alpha_n\} = \sum_{j=1}^{n} (-1)^{n+j} \mathcal{L}_{(\#(\alpha_1 \wedge \ldots \wedge \widehat{\alpha_j} \wedge \ldots \wedge \alpha_n))} \alpha_j - (n-1)\, \mathrm{d}(\Lambda(\alpha_1, \ldots, \alpha_n))$$

for $\alpha_j \in \Omega^1(M)$ $(j = 1, \ldots, n)$ (see [24, 25]).

Taking into account the relation $\mathcal{L}_X = \mathrm{d}i(X) + i(X)\mathrm{d}$, we obtain that $\mathcal{L}_{(fX)} = \mathrm{d}f \wedge i(X) + f\mathcal{L}_X$. Using this fact we prove the following result.

Proposition 4.5.

(i) $\{\mathrm{d}f_1, \ldots, \mathrm{d}f_n\} = \mathrm{d}\{f_1, \ldots, f_n\}$.

(ii) $\{f\alpha_1, \ldots, \alpha_n\} = f\{\alpha_1, \ldots, \alpha_n\} + (-1)^{n+1}(\#(\alpha_2 \wedge \ldots \wedge \alpha_n) f)\alpha_1$.

Since any closed form is locally an exact form, we see that if M is a Nambu–Poisson manifold and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}$ are closed 1-forms then

$$\{\beta_1, \ldots, \beta_{n-1}, \{\alpha_1, \ldots, \alpha_n\}\} = \sum_{i=1}^{n} \{\alpha_1, \ldots, \{\beta_1, \ldots, \beta_{n-1}, \alpha_i\}, \ldots, \alpha_n\}.$$

Also, for a generalized Poisson manifold of even order $2n$, we have

$$\mathrm{Alt}(\{\beta_1, \ldots, \beta_{2n-1}, \{\alpha_1, \ldots, \alpha_{2n}\}\}) = 0$$

if $\beta_1, \ldots, \beta_{2n-1}, \alpha_1, \ldots, \alpha_{2n}$ are closed 1-forms.

Using (6) and proposition 4.5, we deduce the following.

Proposition 4.6. If $P \in \mathcal{V}^k(M)$ then

$$\begin{aligned}
\partial P(\alpha_1, \ldots, \alpha_{k+2n-1}) = & \sum_{1 \leqslant i_1 < \ldots < i_{2n} \leqslant k+2n-1} (-1)^{i_1 + \cdots + i_{2n} + n + 1} \\
& \times P(\{\alpha_{i_1}, \ldots, \alpha_{i_{2n}}\}, \alpha_1, \ldots, \widehat{\alpha_{i_1}} \ldots, \widehat{\alpha_{i_{2n}}}, \ldots, \alpha_{k+2n-1}) \\
& + \sum_{1 \leqslant i_1 < \ldots < i_{2n-1} \leqslant k+2n-1} (-1)^{i_1 + \ldots + i_{2n-1} + n} \#(\alpha_{i_1} \wedge \wedge \ldots \wedge \alpha_{i_{2n-1}}) \\
& \times (P(\alpha_1, \ldots, \widehat{\alpha_{i_1}}, \ldots, \widehat{\alpha_{i_{2n-1}}}, \ldots, \alpha_{k+2n-1}))
\end{aligned}$$

for $\alpha_1, \ldots, \alpha_{k+2n-1} \in \Omega^1(M)$.

Next, we will obtain some relations between the de Rham cohomology and the gP-cohomology.

Let k be an integer, $k \geqslant 1$. Consider the homomorphism of $C^\infty(M, \mathbb{R})$-modules

$$\tilde{\#} : \Omega^k(M) \longrightarrow \mathcal{V}^{k(2n-1)}(M)$$

given by

$$\begin{aligned}
\tilde{\#}(\alpha)(\alpha_1, \ldots, \alpha_{k(2n-1)}) = & \frac{(-1)^k}{k![(2n-1)!]^k} \sum_{\sigma \in \mathfrak{S}_{k(2n-1)}} \varepsilon_\sigma \alpha(\#(\alpha_{\sigma(1)} \wedge \ldots \wedge \alpha_{\sigma(2n-1)}), \ldots, \\
& \#(\alpha_{\sigma((k-1)(2n-1)+1)} \wedge \ldots \wedge \alpha_{\sigma(k(2n-1))}))
\end{aligned} \qquad (7)$$

for $\alpha \in \Omega^k(M)$ and $\alpha_1, \ldots, \alpha_{k(2n-1)} \in \Omega^1(M)$, where $\mathfrak{S}_{k(2n-1)}$ is the group of the permutations of order $k(2n-1)$ and ε_σ is the signature of the permutation σ.

For a C^∞-function f, we define

$$\tilde{\#}(f) = f. \qquad (8)$$

A direct computation, using (2) and (7), shows the following.

Lemma 4.7.

(i) If $\alpha, \alpha_1, \ldots, \alpha_{2n-1}$ are 1-forms on M then

$$\tilde{\#}(\alpha)(\alpha_1, \ldots, \alpha_{2n-1}) = \alpha(\#(\alpha_1 \wedge \ldots \wedge \alpha_{2n-1})).$$

(ii) If $\alpha, \alpha_1, \ldots, \alpha_k$ are 1-forms on M then

$$\tilde{\#}(\alpha_1 \wedge \ldots \wedge \alpha_k) = \tilde{\#}(\alpha_1) \wedge \ldots \wedge \tilde{\#}(\alpha_k).$$

Now, we can prove the following.

Theorem 4.8. Let (M, Λ) be a generalized Poisson manifold of order $2n$ and $\tilde{\#} : \Omega^k(M) \longrightarrow \mathcal{V}^{k(2n-1)}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules given by (7) and (8). Then we have

$$\tilde{\#} \circ \mathrm{d} = -\partial \circ \tilde{\#}.$$

So, $\tilde{\#}$ induces a homomorphism of complexes $\tilde{\#} : (\Omega^*(M), \mathrm{d}) \longrightarrow (\mathcal{V}^{*(2n-1)}(M), -\partial)$, and we have the corresponding homomorphism in cohomology $\tilde{\#} : H^*_{\mathrm{dR}}(M) \longrightarrow H^{*(2n-1)}_{0\,\mathrm{gP}}(M)$.

Proof. From definition 4.2 and lemma 4.7, we deduce

$$\tilde{\#}(\mathrm{d}f) = -\partial f \tag{9}$$

for $f \in C^\infty(M, \mathbb{R})$.

Thus, using (8), (9), proposition 4.3 and lemma 4.7, we obtain

$$\tilde{\#}(\mathrm{d}(f_0 \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k)) = -\partial(\tilde{\#}(f_0 \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k))$$

for $f_0, f_1, \ldots, f_k \in C^\infty(M, \mathbb{R})$.

This completes the proof of our result. □

Remark 4.9. If (M, Λ) is a Poisson manifold, the homomorphism $\tilde{\#} : H^*_{\mathrm{dR}}(M) \longrightarrow H^*_{0\,\mathrm{gP}}(M)$ of theorem 4.8 is just the canonical homomorphism between the de Rham cohomology and the LP-cohomology of M (see [19, 24]).

Next, we will obtain some relations between the gP-cohomology and the canonical homology.

Using that $[[i(P), \mathrm{d}], i(Q)] = i([P, Q])$ we deduce the following.

Proposition 4.10. For $P \in \mathcal{V}^k(M)$ and $\lambda \in \Omega^{k+2n-1}(M)$, we have

$$\langle \lambda, \partial P \rangle - \langle \delta\lambda, P \rangle = -\delta(i(P)\lambda)$$

where $\langle\, , \rangle : \Omega^r(M) \times \mathcal{V}^r(M) \longrightarrow C^\infty(M, \mathbb{R})$ is the duality map defined by

$$\langle \alpha, Q \rangle = i(Q)\alpha.$$

From proposition 4.10, it follows that the duality map induces a natural pairing

$$\langle\, , \rangle : H_k^{j\,\mathrm{can}}(M) \times H^k_{j\,\mathrm{gP}}(M) \longrightarrow H_0^{0\,\mathrm{can}}(M)$$

given by

$$\langle [\lambda], [P] \rangle = [\langle \lambda, P \rangle].$$

Example 4.11.

(i) For a symplectic manifold M of dimension $2m$ we have that (see [6, 19])

$$H^k_{\mathrm{LP}}(M) \cong H^k_{\mathrm{dR}}(M) \cong H^{\mathrm{can}}_{2m-k}(M).$$

(ii) For an almost cosymplectic manifold of dimension $2m+1$ we get that

$$H^k_{\mathrm{LP}}(M) \cong H^{\mathrm{can}}_{2m-k}(M)$$

but they are not isomorphic to the de Rham cohomology (see [10, 17]).

(iii) Let υ_M be a volume form on a compact (connected) manifold M of dimension $2n \geqslant 4$ with associated generalized Poisson tensor Λ. Then, the gP-cohomology groups are the following:

$$\begin{aligned}
&H^0_{0\,\mathrm{gP}}(M) = \{\text{Casimir functions}\} \cong \mathbb{R}\\
&H^{2n-1}_{0\,\mathrm{gP}}(M) = \mathcal{V}^{2n-1}(M)/\{\partial f \,|\, f \in C^\infty(M,\mathbb{R})\}\\
&H^1_{1\,\mathrm{gP}}(M) = \{X \in \mathcal{V}^1(M) | L_X\Lambda = 0\}\\
&H^{2n}_{1\,\mathrm{gP}}(M) = \mathcal{V}^{2n}(M)/\{\partial X | X \in \mathcal{V}^1(M)\} \cong \mathbb{R}\\
&H^k_{j\,\mathrm{gP}}(M) = \mathcal{V}^k(M) \qquad \text{for } k = 2,\ldots,2n-2,\ j \neq 0, 1.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
&H^{0\,\mathrm{can}}_0(M) \cong H^{2n}_{1\,\mathrm{gP}}(M) \qquad H^{0\,\mathrm{can}}_{2n-1}(M) \cong H^1_{1\,\mathrm{gP}}(M)\\
&H^{1\,\mathrm{can}}_1(M) \cong H^{2n-1}_{0\,\mathrm{gP}}(M) \qquad H^{1\,\mathrm{can}}_{2n}(M) \cong H^0_{0\,\mathrm{gP}}(M)\\
&H^{j\,\mathrm{can}}_{2n-k}(M) \cong H^k_{j\,\mathrm{gP}}(M) \qquad \forall j \neq 0, 1, k = 2,\ldots,2n-2.
\end{aligned}$$

(iv) In general, there is no an isomorphism between the generalized canonical homology and the gP-cohomology, even in the Poisson case, as it has been proved in [12].

(v) Let M be a (connected) hyper-Kähler manifold M (see example 3.10). From a direct computation, we deduce that

$$\partial(P) = -2\sum_{i=1}^{3} \partial_i(P) \wedge \Lambda_i$$

where ∂_i denotes the differential operator associated to the Poisson structure Λ_i. However, there is no a simple relation between the gP-cohomology and the de Rham cohomology of M.

(vi) Let (M,ω) be a symplectic $2n$-dimensional manifold with Poisson tensor $\bar{\Lambda}$ and put $\Lambda = \bar{\Lambda} \wedge \bar{\Lambda}$ (see example 3.11). If ∂ (resp. $\bar{\partial}$) is the contravariant derivative for Λ (resp. $\bar{\Lambda}$) we have

$$\partial P = 2\bar{\partial} P \wedge \bar{\Lambda} \qquad \text{for all } P \in \mathcal{V}^k(M).$$

Define now the differential operator

$$\begin{aligned}
&\tilde{\mathrm{d}} : \Omega^k(M) \longrightarrow \Omega^{k+3}(M)\\
&\alpha \longmapsto \tilde{\mathrm{d}}\alpha = 2\mathrm{d}\alpha \wedge \omega.
\end{aligned}$$

Notice that if $L : \Omega^k(M) \longrightarrow \Omega^{k+2}(M)$ is the linear operator defined by $L\alpha = \alpha \wedge \omega$, we get $\tilde{\mathrm{d}}\alpha = 2\mathrm{d}L\alpha$. A direct computation shows that $\tilde{\mathrm{d}}^2 = 0$. Thus, we obtain the following three complexes

$$0 \longrightarrow \Omega^j(M) \xrightarrow{\tilde{d}} \Omega^{j+3}(M) \xrightarrow{\tilde{d}} \Omega^{j+6}(M) \xrightarrow{\tilde{d}} \cdots$$

for $j = 0, 1, 2$. We denote the corresponding cohomology groups by $\tilde{H}^{3i+j}_j(M)$.

If $\bar{\#} : \Omega^k(M) \longrightarrow \mathcal{V}^k(M)$ is the isomorphism defined by $\bar{\Lambda}$ we deduce that $\bar{\#}\tilde{\mathrm{d}} = -\partial\bar{\#}$ and therefore:

$$\tilde{H}_j^{3i+j}(M) \cong H_{j\,\mathrm{gP}}^{3i+j}(M)$$

for $j = 0, 1, 2$. In order to obtain some results on the $\tilde{\mathrm{d}}$-cohomology we will use the following facts (see [18]):

(a) L is injective for $k \leqslant n-1$;

(b) L is surjective for $k \geqslant n-1$.

Hence we have:

$$\tilde{H}_1^1(M) = \{\alpha \in \Omega^1(M) | \mathrm{d}\alpha = 0\} \qquad \text{if } n \geqslant 3$$
$$\tilde{H}_2^2(M) = \{\alpha \in \Omega^2(M) | \mathrm{d}\alpha = 0\} \qquad \text{if } n \geqslant 4.$$

This shows that, in general, the cohomology groups $\tilde{H}_j^{3i+j}(M)$ are not finite dimensional.

Moreover, if $3i + j \leqslant n - 2$ we deduce that the identity transformation $\Omega^{3i+j}(M) \longrightarrow \Omega^{3i+j}(M)$ induces a linear mapping $\tilde{H}_j^{3i+j}(M) \longrightarrow H_{\mathrm{dR}}^{3i+j}(M)$ which is surjective.

On the other hand, if $3i + j \geqslant n + 2$ we obtain that the identity transformation $\Omega^{3i+j}(M) \longrightarrow \Omega^{3i+j}(M)$ induces a linear mapping $H_{\mathrm{dR}}^{3i+j}(M) \longrightarrow \tilde{H}_j^{3i+j}(M)$ which is injective.

5. Canonical homology and gP-cohomology for arbitrary Nambu–Poisson manifolds

In this paper we have introduced and studied the generalized Koszul differential (definition 3.1) and the contravariant exterior differentiation (definition 4.2) for generalized Poisson structures Λ of even order on smooth manifolds. But, the Nambu–Poisson structures are another important structures that generalize the Poisson structures for any order.

As we have seen in [15] the even Nambu–Poisson structures Λ are generalized Poisson (i.e. $[\Lambda, \Lambda] = 0$); moreover, since for any multivector Λ of odd order is always true that $[\Lambda, \Lambda] = 0$, then we can consider the generalized Koszul differential and the contravariant exterior differentiation for Nambu–Poisson structures of any order. In general, for any generalized almost Poisson structure Λ of any order satisfying $[\Lambda, \Lambda] = 0$, it can be defined the generalized Koszul differential and the contravariant exterior differentiation as in definitions 3.1 and 4.2.

Remark 5.1. It should be noticed that in [3] the authors considered almost Poisson tensors of arbitrary order, introducing an integrability condition in the odd order case as a new algebraic relation. Thus, any Nambu–Poisson structure is generalized Poisson, accordingly with their definition.

Remark 5.2. Although for a multivector Λ of odd order the condition $[\Lambda, \Lambda] = 0$ is trivially satisfied, this does not imply that the canonical homology and the gP-cohomology is trivial for these structures, as we will see in the following examples.

Example 5.3 (volume forms). Let υ_M be a volume form in a compact (connected) manifold M of any dimension, then the results of examples 3.4 and 3.9 for the canonical homology and the results of example 4.11 (part (iii)) for the gP-cohomology, are still satisfied.

Example 5.4 (canonical Nambu–Poisson structures). Let us consider the manifold $\mathbb{R}^m$, then the canonical Nambu–Poisson tensor of order n on $\mathbb{R}^m$ is given by

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$$

for coordinates $(x_1, \ldots, x_m)$ on $\mathbb{R}^m$. First, we will calculate the space of Casimir functions:

$$\mathcal{C} = \left\{ f \in C^\infty(\mathbb{R}^m, \mathbb{R}) | \frac{\partial}{\partial x_i}(f) = 0, \text{for all } i = 1, \ldots, n \right\} \cong C^\infty(\mathbb{R}^{m-n}, \mathbb{R}).$$

The canonical homology group of degree m is:

$$H_m^{j\,\mathrm{can}}(\mathbb{R}^m) = \{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m\} \otimes \mathcal{C}$$

with $0 \leqslant j \leqslant n-2$ and $m - j$ a multiple of $n - 1$.

The gP-cohomology groups of degrees $0, 1$ are:

$$H^0_{0\,\mathrm{gP}}(\mathbb{R}^m) = \mathcal{C}$$

$$H^1_{1\,\mathrm{gP}}(\mathbb{R}^m) = \left\{ \sum_{i=1}^n f^i \frac{\partial}{\partial x_i} \middle| \sum_{i=1}^n \frac{\partial}{\partial x_i}(f^i) = 0 \right\} \oplus \left(\left\langle \frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_m} \right\rangle_{\mathbb{R}} \otimes \mathcal{C} \right).$$

Now, for $n = 3$ we will calculate the gP-cohomology group of degree 2:

$$H^2_{0\,\mathrm{gP}}(\mathbb{R}^m) = \frac{\Lambda^2\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)}{\{i(\mathrm{d}f)\Lambda | f \in C^\infty(\mathbb{R}^m, \mathbb{R})\}} \oplus \left\{ \sum_{j>3, i=1}^3 f^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} | \sum_{i=1}^3 \frac{\partial}{\partial x_i}(f^{ij}) = 0 \right\}$$
$$\oplus \left(\Lambda^2_{\mathbb{R}} \left(\frac{\partial}{\partial x_4}, \ldots, \frac{\partial}{\partial x_m} \right) \otimes \mathcal{C} \right)$$

and, for $n > 3$:

$$H^2_{2\,\mathrm{gP}}(\mathbb{R}^m) = \Lambda^2\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \oplus \left\{ \sum_{j>n, i=1}^n f^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \middle| \sum_{i=1}^n \frac{\partial}{\partial x_i}(f^{ij}) = 0 \right\}$$
$$\oplus \Lambda^2_{\mathbb{R}} \left(\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_m} \right) \otimes \mathcal{C}.$$

Example 5.5 (Nambu–Poisson manifolds). From the fact that the Hamiltonian vector fields are infinitesimal automorphisms of the Nambu–Poisson structure [15], we have that the space of Hamiltonian vector fields of a Nambu–Poisson manifold (of degree n) is a subspace of the gP-cohomology group of degree 1, that is, $\{X_{f_1 \ldots f_{n-1}} | f_i \in C^\infty(M, \mathbb{R})\} \subseteq H^1_{1\,\mathrm{gP}}(M)$. In general, this inclusion is strict as we have seen in example 5.4.

Acknowledgments

This work was partially supported through grants DGICYT (Spain) projects PB94-0106, PB94-0633-C02-02, Departamento de Educación, Universidades e Investigación del Gobierno Vasco, Consejería de Educación y Cultura de la Comunidad de Madrid, and Universidad de La Laguna. We acknowledge the referees for their valuable suggestions and remarks.

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